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Stealths on Anisotropic Holographic Backgrounds

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Abstract: In this paper, we are interested in exploring the existence of stealth configurations on anisotropic backgrounds playing a prominent role in the non-relativistic version of the gauge/gravity correspondence. By stealth configuration, we mean a nontrivial scalar field nonminimally coupled to gravity whose energy-momentum tensor evaluated on the anisotropic background vanishes identically. In the case of a Lifshitz spacetime with a nontrivial dynamical exponent z, we spotlight the role played by the anisotropy to establish the holographic character of the stealth configurations, i.e. the scalar field is shown to only depend on the radial holographic direction. This configuration which turns out to be massless and without integration constants is possible for a unique value of the nonminimal coupling parameter. Then, using a simple conformal argument, we map this configuration into a stealth solution defined on the so-called hyperscaling violation metric which is conformally related to the Lifshitz spacetime. This holographic configuration obtained through a conformal mapping constitutes only a particular class within the stealth solutions defined on the hyperscaling violation as it is shown by deriving the most general stealth configurations. The case of the Schrödinger background is also exhaustively analyzed and we establish that the presence of the null direction makes their stealth configurations not necessarily holographic in general and characterized by a self-interacting behavior. Finally, for completeness we also study the stealth configurations on the Schrödinger inspired hyperscaling violation spacetimes.

Contents

1. Introduction

The fundamental tenet of General Relativity lies in the fact that gravity is a manifestation of the curvature of spacetime produced by the presence of matter sources. This phenomenon is encoded in the equations proposed by Einstein a century ago that relate a gravity tensor in the left hand side, which only depends on the metric, to the energy-momentum tensor of the matter source in the right hand side. Nevertheless, one can ask the following question: can there exist background metrics and nontrivial matter sources such that both sides of Einstein equations vanish identically? This question can be reformulated as follows: for a fixed gravitational background presenting some physical interest, does a nontrivial matter source exist such that its energy-momentum vanishes once evaluated on this background? These kind of solutions of Einstein equations have been dubbed stealth configurations and are characterized by their lack of back-reaction on the gravitational field causing an impossibility of curving the underlying geometry. Such configurations have been derived previously using scalar fields nonminimally coupled to gravity on the BTZ black hole [1] in three dimensions [2], and on higher-dimensional Minkowski [3] and (A)dS spacetimes [4]. Concretely for the

conformal coupling, they have been also exhibited on any homogeneous and isotropic universe [5].

In this work, we explore whether nonminimally coupled (and possibly self-interacting) scalar fields may still be a good laboratory to define stealth configurations on spacetimes playing a prominent role in the gauge/gravity correspondence [6]. The paradigmatic example as gravitational dual is the AdS space. The existence of the stealth in this case has been established in [4]. Other interesting gravitational duals are those occurring in the non-relativistic versions of the gauge/gravity correspondence. Significant examples are those where anisotropic scalings play an outstanding role such as the Lifshitz spacetimes [7] or their generalizations exhibiting hyperscaling violation [8]. Historically, the first example of non-relativistic holography was provided with the Schrödinger background, see [9, 10]. However, the presence of a (compact) null direction required to ensure the Galilean invariance makes its holographic interpretation a subject of debate.

The relevance of stealth configurations for gauge/gravity duality lies in the fact that their fluctuations are not expected to be stealth themselves.[1] Correspondingly, the perturbations of the gravitational duals could be modified by the presence of the stealth, and since this weak field limit is in correspondence to the strongly correlated regime of the dual quantum theory, the associated holographic predictions could be potentially modified. In this paper, we definitely establish the existence of stealth configurations on all the anisotropic backgrounds studied in the current literature, namely the Lifshitz and Schrödinger backgrounds as well as for their hyperscaling violation generalizations, and spotlight their similitude and their differences. This paves the way to initiate an ensuing program exploring the holographic consequences of the existence of the stealths and their nontrivial perturbations in the non-relativistic version of gauge/gravity duality.

The action we choose to model stealth configurations is the one of a self-interacting scalar field Φ nonminimally coupled to gravity and parameterized in terms of the nonminimal coupling parameter ξ as

$$S_{\xi}[\Phi, g_{\mu\nu}] = \int d^D x \sqrt{-g}\left(-\frac{1}{2}\partial_{\mu}\Phi\, \partial^{\mu}\Phi - \frac{1}{2}\xi R\, \Phi^2 - U(\Phi)\right), \tag{1.1}$$

where R stands for the scalar curvature and $U(\Phi)$ represents a possible self-interaction potential. As said before, a stealth configuration is composed by two ingredients: the given spacetime background (which, in our case, will be the Lifshitz, the Schrödinger or their hyperscaling generalizations) and the nontrivial field (given by the nonminimally coupled scalar field) whose energy-momentum tensor evaluated on the background vanishes. In this sense, the related constraints can also be understood as if both ingredients correspond to extrema of the matter action describing just the field. In fact, fixing the background a priori, the related conditions become only constraints for the dynamical field. In our case, the conditions

[1] We thank C. Terrero-Escalante for pointing out this non trivial issue.

defining the stealth, which also correspond to the variations of the action (1.1), are given by

$$T_{\alpha\beta} = \partial_\alpha\Phi\,\partial_\beta\Phi - g_{\alpha\beta}\left(\frac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi + U(\Phi)\right) + \xi\left(g_{\alpha\beta}\Box - \nabla_\alpha\nabla_\beta + G_{\alpha\beta}\right)\Phi^2 = 0, \tag{1.2a}$$

$$\Box\Phi - \xi R\Phi - \frac{dU(\Phi)}{d\Phi} = 0. \tag{1.2b}$$

In fact, it is sufficient to solve only the first constraints (1.2a), since the fulfillment of the field equation (1.2b) for a nontrivial field is warranted from the conservation of the energy-momentum tensor due to the diffeomorphism invariance of action (1.1).

The paper is organized as follows. We start with a brief introduction on non-relativistic holographic backgrounds in Sec. 2. After that we prove in Sec. 3 that Lifshitz spacetimes support stealth configurations by solving the stealth constraints on the Lifshitz background in full generality. We emphasize that a straightforward consequence of its anisotropy is that, contrary to the AdS case [4], the Lifshitz stealth is not only stationary but must be also strictly holographic in nature in the sense that it only admits a dependence on the holographic direction. Other interesting features which differ radically from the isotropic cases are that the nonminimal coupling parameter takes a fixed value parameterized in terms of the corresponding dynamical exponent for any dimension, and these anisotropic configurations must be free of any self-interaction. In Sec. 4, we exploit the conformal relation between the Lifshitz and hyperscaling violation metrics and we extend the stealth configuration to the last spacetime using conformal arguments in arbitrary dimension. Later on, we show that these configurations constitute only a particular class of the possible stealths on the hyperscaling violation metric by deriving its most general stealth solutions. We exhibit special values of the exponents for which the stealth depends on the transverse directions in addition to the holographic coordinate. In Sec. 5 the Schrödinger case is analyzed in detail. The presence of the null coordinate makes the Schrödinger stealth configuration not necessarily holographic. Finally, we also analyze the hyperscaling violation case derived from the Schrödinger metric. The last section is devoted to our conclusions.

2. Non-relativistic holographic backgrounds

In condensed matter physics, a quantum phase transition occurs between two different phases at zero temperature. At this critical point, the system becomes invariant under a scaling symmetry with eventually different weights between space and time,

$$t \mapsto \lambda^z\, t, \qquad \vec{x} \mapsto \lambda\vec{x}, \tag{2.1}$$

where the relative weight z is called the dynamical critical exponent. The quantum critical point can be very useful to have a good comprehension of the entire phase diagram, and hence a good analysis of this point can be of particular interest in order to compute transport coefficients, like the electrical conductivity in the case of superfluid-insulator phase transition or some thermal quantities. In general, these quantities are difficult to compute because the

systems are usually strongly coupled. The AdS/CFT correspondence, valid for $z = 1$, has been proved to be a very promising tool for studying strongly coupled systems by mapping them into the weak regime of classical theories of gravity and establishing a dictionary between both systems [6]. However, other values of z are present experimentally in particular for different condensed matter phenomena. This is one of the main motivations behind the recent interest of extending the AdS/CFT correspondence to non-relativistic physics and to condensed matter applications. In this perspective, there are two symmetry groups playing an important role: the Schrödinger group which may be viewed as the non-relativistic cousin of the conformal group and the Lifshitz group which is characterized by an anisotropic scaling symmetry but without the Galilean boosts of the first example. For this latter case, the gravity dual metric is referred to as the Lifshitz metric [7] and is given in D dimensions by

$$ds_{\mathrm{L}}^2 = -\frac{r^{2z}}{l^{2z}}dt^2 + \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}d\vec{x}^2, \tag{2.2}$$

where $\vec{x} = (x^1, \ldots, x^{D-2})$. Indeed, it is simple to check that the dynamical scalings (2.1) supplemented with an additional scaling in the holographic direction, $r \mapsto \lambda^{-1}r$, act as an isometry for the metric (2.2). In the last years, there has been an intensive activity looking for asymptotically Lifshitz black holes; some examples are given in [11, 12, 13, 14] and references therein. Holographically, these solutions should describe the finite temperature behavior of the related non-relativistic systems. Here $z = 1$ describes the isotropic paradigm since the metric becomes the one of AdS spacetime, whose stealth configurations were studied in [4]. Another special situation, although holographically less motivated, is for $z = 0$ since in this case the resulting Lifshitz background becomes conformally flat. Nevertheless, as shown below, this last feature induces nontrivial consequences regarding the existence of conformal stealths.

More recently, there has been some interest in extending this kind of metrics (2.2) by introducing an additional parameter, the hyperscaling violation exponent θ, such that the scaling transformations do not act as an isometry but rather like a conformal transformation. These metrics, referred to as hyperscaling violation metrics, are described by the following line element [8]

$$ds_{\mathrm{H}}^2 = \left(\frac{l}{r}\right)^{\frac{2\theta}{D-2}} \left(-\frac{r^{2z}}{l^{2z}}dt^2 + \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}d\vec{x}^2\right), \tag{2.3}$$

and transform as $ds_{\mathrm{H}}^2 \mapsto \lambda^{\frac{2\theta}{D-2}} ds_{\mathrm{H}}^2$ under the scaling (2.1) supplemented with the holographic scaling $r \mapsto \lambda^{-1}r$. Notice that this metric is conformally related to the Lifshitz one (2.2) which is recovered in the limiting case $\theta = 0$. As in the Lifshitz case, there is a physical interest in looking for black holes whose asymptotic behavior coincides with the hyperscaling violation metric, see e.g. [15]. For $\theta = D - 2$, this spacetime just becomes the Minkowski one if $z = 0$ or $z = 1$, whose stealth configurations were originally reported in [3].

As mentioned before, the first attempt to extend the ideas of the AdS/CFT correspondence to non-relativistic physics was done in the context of the symmetry group of the

Schrödinger equation for the free particle, which can be viewed as the non-relativistic cousin of the conformal group. The gravity dual metric in this case, defining the Schrödinger spacetime [9, 10], is given in D dimensions by

$$ds^2_{\mathrm{S}} = \frac{l^2}{y^2}\left[-\left(\frac{l}{y}\right)^{2(z-1)} du^2 - 2du dv + dy^2 + d\vec{x}^2\right]. \tag{2.4}$$

where $\vec{x}$ is now a $(D-3)$-dimensional vector. A geometrical derivation of this metric using the conformal invariance of the Schrödinger equation can be found in [17]. We have intentionally changed the notations of the line element to be in perfect accordance with those in Ref. [18, 19] that will be our guiding principle to derive the general stealth configuration on the Schrödinger background (2.4). Indeed, the role of non-relativistic time in the dynamical scaling (2.1) is now played by the retarded time u while the holographic coordinate r is replaced by y^{-1} such that the boundary is now located at $y = 0$. This class of metrics is invariant not only under the anisotropic scaling determined by the exponent z, but also under Galilean transformations. For $z = 2$, the metric enjoys the *full* Schrödinger symmetry, see for example [20] for an account of all these symmetries. The particular case $z = 1$ is maximally symmetric since we again recover the AdS metric written in light-cone coordinates. Another case that will deserve a special attention is for $z = 1/2$, since the metric inside the squared brackets becomes the flat spacetime, i.e. the Schrödinger background becomes conformally flat. In contrast with the previous cases, the presence of the null coordinate v required by the Galileo boosts, suggests that this metric will possibly describe non-relativistic quantum theory in dimension $D-2$ and not in co-dimension one. This remark together with the fact that this additional coordinate represents a compact direction makes its holographic interpretation unclear, as previously stated. However, we shall also study this case in order to be exhaustive in the comprehension of stealth configurations on anisotropic backgrounds.

Finally, and consistently with our strategy of completeness, it is also possible to define a Schrödinger inspired background with hyperscaling violation [21]. The relevant line element in this case is given by

$$ds^2_{\mathrm{HS}} = \left(\frac{y}{l}\right)^{\frac{2(\theta-D+2)}{D-2}}\left[-\left(\frac{l}{y}\right)^{2(z-1)} du^2 - 2du dv + dy^2 + d\vec{x}^2\right]. \tag{2.5}$$

For $\theta = D-2$, this metric describes *pp*-waves, and if additionally $z = 1$ or $z = 1/2$ we recover the flat spacetime, whose stealths were studied in [3]. These metrics will end our exploration concerning the existence of stealths on anisotropic backgrounds.

3. Stealths on Lifshitz backgrounds

Here, we will solve the stealth constraints (1.2) on the Lifshitz background (2.2) in full generality, that is, without any extra assumption. We start by considering the generic Lifshitz case where the dynamical critical exponents $z \neq 1$ and $z \neq 0$ in the first subsection. In

particular, we will establish the holographic nature of the Lifshitz stealth configuration for those nontrivial values of the dynamical exponent. The isotropic case, $z = 1$, was already addressed in [4] and the vanishing case, $z = 0$, is poorly motivated from the holographic point of view. However, for completeness we will also consider this case in the second subsection.

3.1 Nontrivial dynamical exponents: Holographic stealths

For a nonminimal coupling parameter $\xi \neq 1/4$, it is useful to redefine the scalar field as

$$\Phi = \frac{1}{\sigma^{2\xi/(1-4\xi)}}, \tag{3.1}$$

where $\sigma = \sigma(x^\mu)$ is a local function depending on all coordinates. The case $\xi = 1/4$ will be analyzed at the end of the subsection. Just for completeness, we mention that stealth configurations are not allowed in the minimal case, $\xi = 0$, independently of the background. Using the above redefinition, the off-diagonal components of the energy-momentum tensor (1.2a) along the time coordinate give rise to the following constraints

$$T_{\mu t} = \frac{(2\xi)^2}{1-4\xi}\left(\frac{r}{l}\right)^z \frac{\Phi^2}{\sigma}\,\partial^2_{\mu t}\left[\left(\frac{l}{r}\right)^z \sigma\right] = 0, \qquad \mu \neq t, \tag{3.2}$$

which in turn implies that

$$\sigma(t, r, x^i) = \left(\frac{r}{l}\right)^z T(t) + \hat{\sigma}(r, x^i), \tag{3.3}$$

where T (resp. $\hat{\sigma}$) is an arbitrary function of t (resp. of r and the spatial coordinates). The remaining off-diagonal constraints are expressed by

$$T_{ri} = \frac{(2\xi)^2}{1-4\xi}\frac{r}{l}\frac{\Phi^2}{\sigma}\,\partial^2_{ri}\left(\frac{l}{r}\hat{\sigma}\right) = 0, \tag{3.4a}$$

$$T_{ij} = \frac{(2\xi)^2}{1-4\xi}\frac{r}{l}\frac{\Phi^2}{\sigma}\,\partial^2_{ij}\left(\frac{l}{r}\hat{\sigma}\right) = 0, \quad i \neq j, \tag{3.4b}$$

and those latter impose that $l\hat{\sigma}/r$ is totally separable in sum with respect to all its dependencies. This takes into account all the $D(D-1)/2$ off-diagonal constraints and permits to conclude that the function σ satisfies the following separability

$$\sigma(t, r, x^i) = \left(\frac{r}{l}\right)^z T(t) + \frac{l}{r}H(r) + \frac{r}{l}\left[X^1(x^1) + \cdots + X^{D-2}(x^{D-2})\right], \tag{3.5}$$

where H is a function of the holographic coordinate r, and where each function X^i only depends on the planar coordinate x^i. Moreover, we emphasize that the functions involved in the separability are not uniquely defined. In fact, they are determined modulo the following transformations

$$(X^i, X^j) \mapsto (X^i - C^{ij}, X^j + C^{ij}), \tag{3.6a}$$

$$(X^i, H) \mapsto (X^i - C^i, H + C^i\, r^2/l^2), \tag{3.6b}$$

$$(T, H) \mapsto (T - C^0, H + C^0\, r^{z+1}/l^{z+1}), \tag{3.6c}$$

where C^{ij}, C^i and C^0 are arbitrary constants. These residual symmetries of the separability ansatz (3.5) will be useful in the deduction of the holographic behavior of the stealth.

Let us now consider the diagonal stealth constraints. We start by analyzing the difference between two different spatial components

$$T_{(i)}{}^{(i)} - T_{(j)}{}^{(j)} = \frac{(2\xi)^2}{1-4\xi}\frac{l}{r}\frac{\Phi^2}{\sigma}\left(\frac{\mathrm{d}^2X^i}{\mathrm{d}(x^i)^2} - \frac{\mathrm{d}^2X^j}{\mathrm{d}(x^j)^2}\right) = 0, \quad i \neq j, \tag{3.7}$$

where repeated indices between parenthesis mean that there is no sum for those indices. We infer from these $(D-3)$ requirements that

$$\frac{\mathrm{d}^2X^1}{\mathrm{d}(x^1)^2} = \cdots = \frac{\mathrm{d}^2X^i}{\mathrm{d}(x^i)^2} = \cdots = \frac{\mathrm{d}^2X^{D-2}}{\mathrm{d}(x^{D-2})^2} = \text{const.} \tag{3.8}$$

Using the above conditions together with

$$\partial_i\left(\frac{\sigma\left(T_{(j)}{}^{(j)} - T_r{}^r\right)}{\Phi^2}\right) = \frac{z(z-1)\xi\, r}{l^3}\frac{\mathrm{d}X^i}{\mathrm{d}x^i} = 0, \tag{3.9}$$

we conclude that since $z \neq 0$ and $z \neq 1$ each function X^i is a constant that can be chosen to be zero without loss of generality. This can be seen easily by redefining appropriately the holographic dependence $H(r)$ according to the residual symmetry (3.6b) of the separability ansatz (3.5). Something similar can be deduced for the temporal dependence from the condition

$$\partial_t\left(\frac{\sigma\left(T_t{}^t - T_r{}^r\right)}{\Phi^2}\right) = \frac{(D-2)(z-1)\,\xi}{l^2}\frac{\mathrm{d}T}{\mathrm{d}t}\left(\frac{r}{l}\right)^z + \frac{(2\xi)^2}{(1-4\xi)}\frac{\mathrm{d}^3T}{\mathrm{d}t^3}\left(\frac{r}{l}\right)^{-z} = 0, \tag{3.10}$$

where the coefficients of the different powers of the holographic coordinate r must vanish independently since the dynamical exponent is non-vanishing ($z \neq 0$). As a direct consequence, the function T is also a constant whose value can be taken as zero redefining again the function $H(r)$, but using now the residual symmetry (3.6c). Hence, we conclude that the stealth configuration defined on a Lifshitz spacetime depends only on the holographic coordinate r. We would like to stress that the situation is clearly different in the $z=1$ isotropic-relativistic AdS case and in the trivial case $z=0$ as it can be seen from the expressions (3.9) and (3.10). This makes evident that the generic nontrivial anisotropy of Lifshitz spacetimes is the responsible for the holographic behavior of the stealth scalar field.

The holographic dependence can be additionally fixed by considering the following stealth constraints

$$T_t{}^t - T_{(i)}{}^{(i)} = \frac{(z-1)\,\xi}{(1-4\xi)\,l\,r}\frac{\Phi^2}{\sigma}\left(4\xi r\frac{\mathrm{d}H}{\mathrm{d}r} + \left[4(z+D-3)\xi - z - D + 2\right]H\right) = 0, \tag{3.11a}$$

$$T_r{}^r - T_t{}^t = \frac{\xi}{(1-4\xi)\,l\,r}\frac{\Phi^2}{\sigma}\left(4\xi r^2\frac{\mathrm{d}^2H}{\mathrm{d}r^2} - 4\xi(z+1)\,r\frac{\mathrm{d}H}{\mathrm{d}r} - \left\{4\left[(D-3)z - D + 1\right]\xi - (D-2)(z-1)\right\}H\right) = 0. \tag{3.11b}$$

The first equation imposes a power-law behavior for the holographic dependence while the second one restricts the value of the nonminimal coupling parameter ξ in terms of the dynamical exponent z for any dimension D yielding to

$$\xi_{\mathrm{L}} \equiv \frac{(z+D-2)^2}{4\left[(z+D-2)^2+z^2+D-2\right]}, \tag{3.12a}$$

$$\Phi(r) = \Phi_0 \left(\frac{l}{r}\right)^{\frac{z+D-2}{2}}. \tag{3.12b}$$

Here, the constant Φ_0 can be tuned arbitrarily using the scaling symmetry of Lifshitz backgrounds. It remains to determine the allowed self-interacting potential. However, evaluating any component of the energy-momentum tensor using the expressions given by (3.12) it is easy to see that $T_{\mu}^{\ \nu} = -U(\Phi)\delta_{\mu}^{\ \nu} = 0$, and hence only massless free stealth configurations are allowed on the Lifshitz background.

Many comments can be made concerning this solution. First, it is interesting to note that the nonminimal coupling parameter is appropriately parameterized in each dimension D in terms of the dynamical critical exponent z. This situation is quite different from the stealth solutions on Minkowski or (A)dS backgrounds since in these cases, the stealths are allowed for any value of the nonminimal coupling parameter. We also stress that the anisotropy is directly responsible of this fixing, since for the $z = 1$ isotropic-relativistic AdS case the two constraints (3.11) reduce to a single one, and this explains the unrestricted behavior of the nonminimal coupling parameter for isotropic backgrounds. Finally, we have seen that in the nontrivial Lifshitz case, also in contrast with the (A)dS or Minkowski cases, only massless free configurations are allowed.

Note that from the expression of the allowed nonminimal coupling parameter (3.12a), this latter is bounded from above as $\xi_{\mathrm{L}} < 1/4$; that is stealth configurations are not possible for $\xi > 1/4$ and also for the limiting case $\xi = 1/4$. Indeed, in this limiting case, the appropriate redefinition is given by $\sqrt{\kappa}\,\Phi(x^{\mu}) = \exp\left[\sigma(x^{\mu})\right]$, and it is easy to prove along the same lines as before that σ satisfies similar constraints than those in the generic case implying first the separability (3.5) and subsequently the strictly holographic dependence. We also end with two restrictions similar to those obtained in (3.11) with the difference that they are not compatible in the present case.

3.2 Conformally flat dynamical exponent: Stealths overflying Lifshitz

As was previously emphasized, the value $z = 0$ is poorly motivated from the point of view of holographic applications. However, it presents some interesting features related with the observation that Lifshitz spacetime is conformally flat for a vanishing dynamical exponent,

which is manifest after rewriting the metric according to

$$\begin{aligned} ds_{\mathrm{L}}^2 &= -dt^2 + \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}d\vec{x}^2 \\ &= \frac{r^2}{l^2}\left\{-\left[d\left(\frac{l^2}{r}\sinh\frac{t}{l}\right)\right]^2 + \left[d\left(\frac{l^2}{r}\cosh\frac{t}{l}\right)\right]^2 + d\vec{x}^2\right\} \equiv \Omega^2\eta_{\mu\nu}d\bar{x}^\mu d\bar{x}^\nu, \end{aligned} \tag{3.13}$$

where $\{\bar{x}^\mu\}$ are the standard cartesian coordinates of flat spacetime. Moreover, for $z=0$ the Lifshitz nonminimal coupling (3.12a) just becomes the conformal one, $\xi_{\mathrm{L}} = \xi_D$, defined by

$$\xi_D \equiv \frac{D-2}{4(D-1)}. \tag{3.14}$$

Only for the conformal coupling, and for a self-interaction potential specified as $U(\Phi) \propto \Phi^{\frac{2D}{D-2}}$, the action (1.1) becomes invariant under conformal transformations (see Subsec. 4.1 for precise definitions). Any symmetry of the action (1.1) is a symmetry of the stealth constraints (1.2). Consequently, for the conformal coupling and the conformal potential the stealth constraints (1.2) are conformally invariant and any stealth solution defines a whole conformal class of stealth solutions. Hence, due to the conformal flatness of the Lifshitz spacetime for $z=0$, which is achieved for the conformal factor defined in (3.13), we can build for $\xi = \xi_D$ a conformal stealth on this background, Φ_{L}, just by performing a conformal transformation to the conformal stealth existing on the flat spacetime, Φ_{F}, and studied in [3], namely,

$$\begin{aligned} \Phi_{\mathrm{L}}(x^\mu) &= \Omega^{-(D-2)/2}\,\Phi_{\mathrm{F}}(\bar{x}^\mu) = \left[\Omega\left(\frac{\alpha}{2}\eta_{\mu\nu}\bar{x}^\mu\bar{x}^\mu + k_\mu\bar{x}^\mu + \sigma_0\right)\right]^{-(D-2)/2} \\ &= \left(\frac{r}{l}\left\{\frac{\alpha}{2}\left[-\left(\frac{l^2}{r}\sinh\frac{t}{l}\right)^2 + \left(\frac{l^2}{r}\cosh\frac{t}{l}\right)^2 + \vec{x}^2\right] + k_t\left(\frac{l^2}{r}\sinh\frac{t}{l}\right) + k_r\left(\frac{l^2}{r}\cosh\frac{t}{l}\right)\right.\right. \\ &\qquad \left.\left. + \vec{k}\cdot\vec{x} + \sigma_0\right\}\right)^{-(D-2)/2} \\ &= \left\{lk_t\sinh\frac{t}{l} + lk_r\cosh\frac{t}{l} + \frac{r}{l}\left[\frac{\alpha}{2}\left(\frac{l^4}{r^2} + \vec{x}^2\right) + \vec{k}\cdot\vec{x} + \sigma_0\right]\right\}^{-(D-2)/2}, \end{aligned} \tag{3.15a}$$

where the conformally invariant potential is determined in the same line as in flat spacetime

$$U(\Phi) = \frac{(D-2)^2}{8}\lambda\Phi^{\frac{2D}{D-2}}, \qquad \lambda = -k_t^2 + k_r^2 + \vec{k}^2 - 2\alpha\sigma_0. \tag{3.15b}$$

This is exactly the same result that is obtained by explicitly integrating the stealth constraints for $z=0$ and $\xi = \xi_D$. We found it more useful to present the above argument than the explicit integration because it makes the origin of this configuration clear. Here the constant k_r can be eliminated using the time translation invariance of Lifshitz spacetime, if $\alpha \neq 0$ the same can be done for $\vec{k}$ by means of space translations and the constant α itself can be tuned to any value with the help of the scaling symmetry. As a consequence the solution presents a

single independent integration constant since the other one is determined by the conformal coupling constant. A similar conclusion is achieved for $\alpha = 0$ using now spatial rotations and the scaling. This integration constant is a result of conformal symmetry as it occurs in flat spacetime [3].

For $\alpha = 0 = k_\mu$ we recover the holographic stealth (3.12) evaluated at $z = 0$. The above conformal configuration contains in contrast a homogeneous subclass when $\alpha = 0 = \vec{k} = \sigma_0$, free of any integration constants. It is possible to show that this exclusively time-dependent solution can be generalized to any value of the nonminimal coupling parameter ξ. In fact, for $\xi \neq \xi_D$ no other behavior than the homogeneous one is possible, giving as result

$$\Phi(t) = \left(\frac{k_t}{\omega}\sinh(\omega t) + l k_r \cosh(\omega t)\right)^{-\frac{2\xi}{1-4\xi}}, \qquad \omega^2 = \frac{(D-2)(1-4\xi)}{4\xi l^2}, \tag{3.16a}$$

$$U(\Phi) = \frac{2\,\xi\Phi^2}{1-4\xi}\left(\frac{\xi\lambda\Phi^{\frac{1-4\xi}{\xi}}}{1-4\xi} - \frac{(D-1)\,(D-2)(\xi-\xi_D)}{l^2}\right), \qquad \lambda = -k_t^2 + l^2\omega^2 k_r^2. \tag{3.16b}$$

Notice that for $\xi = \xi_D$ we consistently recover the homogeneous version of the conformal solution (3.15).

The case $\xi = 1/4$ is excluded from the previous analysis since it is a priori not covered by the redefinition (3.1). However, it is possible to show the existence of a stealth solution in this case that remarkably can be obtained as a nontrivial limit of the above solution. We shall explain the involved procedure in a general setting, since it has the potential to be applied in other contexts. We start reconsidering the redefinition (3.1) by introducing explicitly the dimension of the scalar field as $\sqrt{\kappa}\,\Phi = \sigma^{-2\xi/(1-4\xi)}$, being κ the Einstein constant, in order to have a dimensionless redefined function σ. The solution for σ will be exactly the same, with the difference that the integration constants must have now the proper dimensions that make σ dimensionless; in the present case the constants k_t and k_r in (3.16) will change to have dimensions of inverse length. The remaining procedure is simple, if after possibly redefining the integration constants we find that the following limit is well-behaved

$$\lim_{\xi\to 1/4} \frac{2\xi(1-\sigma(x^\mu))}{1-4\xi} \equiv \hat{\sigma}(x^\mu), \tag{3.17}$$

the configuration for $\xi = 1/4$ can be obtained as

$$\begin{aligned}
\Phi &= \lim_{\xi\to 1/4} \frac{1}{\sqrt{\kappa}}\,\sigma^{-2\xi/(1-4\xi)} \\
&= \lim_{\xi\to 1/4} \frac{1}{\sqrt{\kappa}}\left(1 - \frac{1-4\xi}{2\xi}\hat{\sigma} + O\left((1-4\xi)^2\right)\right)^{-2\xi/(1-4\xi)} \\
&= \frac{1}{\sqrt{\kappa}}\,e^{\hat{\sigma}},
\end{aligned} \tag{3.18}$$

where we have used the limit definition of the exponential function $e^x \equiv \lim_{m\to 0}(1+mx)^{1/m}$. For the solution (3.16) the condition (3.17) is achieved after redefining the integration con-

stants by

$$k_t = -\frac{1-4\xi}{2\xi}\hat{k}_t, \qquad k_r = \frac{1}{l}\left(1 - \frac{1-4\xi}{2\xi}\hat{\sigma}_0\right), \tag{3.19}$$

which gives

$$\hat{\sigma}(t) = -\frac{D-2}{4}\frac{t^2}{l^2} + \hat{k}_t t + \hat{\sigma}_0. \tag{3.20}$$

This procedure must be also consistent when applied to the supporting self-interactions, i.e. taking the nontrivial limit must produce a well-behaved result. For the potential (3.16b) the limit resulting from the redefinitions (3.19), after considering that the coupling constant is now related to the integration constants by $\lambda = \kappa^{(1-4\xi)/(2\xi)}(-k_t^2 + l^2\omega^2 k_r^2)$, is the following

$$\begin{aligned} U(\Phi) &= \lim_{\xi\to 1/4} \frac{(D-2)\Phi^2}{8l^2}\left\{\frac{4\xi}{1-4\xi}\left[\left(\frac{\Phi}{\Phi_0}\right)^{(1-4\xi)/\xi} - 1\right] + D - 1 + O\left(1-4\xi\right)\right\} \\ &= \frac{(D-2)\Phi^2}{8l^2}\left[4\ln\left(\frac{\Phi}{\Phi_0}\right) + D - 1\right], \end{aligned} \tag{3.21}$$

where the coupling constant is given by $\sqrt{\kappa}\,\Phi_0 = \exp[l^2\hat{k}_t^2/(D-2) + \hat{\sigma}_0]$ and additionally we have used the limit definition of the logarithmic function $\ln x \equiv \lim_{m\to 0}(x^m - 1)/m$. The configuration (3.18), (3.20) with the self-interaction (3.21) is precisely the stealth solution resulting from integrating explicitly the stealth constraints for $\xi = 1/4$ when Lifshitz spacetime has a vanishing dynamical exponent. This exhausts all the nonminimally coupled scalar stealths allowed to exist on the Lifshitz background. In the next section we continue with a similar search on hyperscaling violating backgrounds.

4. Stealths in presence of hyperscaling violation

We now look for the existence of stealth configurations defined on hyperscaling violation metric (2.3). Since this background is conformally related to the Lifshitz spacetime (2.2), we first use a conformal argument in order to obtain hyperscaling violation stealth solutions from those derived in the Lifshitz case in Subsec. 4.1. We will see later that the configuration obtained through the conformal mapping represents only a particular class of the stealth configurations on hyperscaling violation metric by deriving the most general solution in Subsec. 4.2. The cases associated to the nonminimal coupling $\xi = 1/4$ are separately analyzed in Subsec. 4.3.

4.1 Holographic branch: Conformal map from Lifshitz

In the standard AdS/CFT correspondence, it is well-known that the AdS metric solves the Einstein equations with a negative cosmological constant. In contrast, in order to support Lifshitz spacetimes (2.2) the vacuum Einstein equations are not enough and they require the introduction of some matter source, e.g. [11, 14], or a more radical approach consists in considering higher order gravity theories as done in Refs. [12, 13]. In the case of the

hyperscaling violation metric (2.3), a simple computation shows that this spacetime is solution of the vacuum Einstein equations [16], without cosmological constant, provided the dynamical and hyperscaling violation exponents are fixed in term of the dimension D as

$$z=\frac{2(D-2)}{D-3},\qquad \theta=\frac{(D-1)(D-2)}{D-3}. \tag{4.1}$$

Here, we first re-derive this result by an indirect method. In fact, we will establish a correspondence between the holographic stealth configuration (3.12) on the Lifshitz spacetime (2.2) defined by a free massless nonminimally coupled scalar field with a solution of the vacuum Einstein equations in the specific case where the nonminimal coupling is given by the conformal one ξ_D defined in (3.14). Indeed, it is well-known that the action (1.1) with $\xi=\xi_D$ and without potential is conformally invariant. More precisely, under the conformal transformation

$$\bar{g}_{\mu\nu}=\Omega^2\,g_{\mu\nu},\qquad \bar{\Phi}=\Omega^{-(D-2)/2}\,\Phi, \tag{4.2}$$

where the conformal factor $\Omega=\Omega(x)$ is any local function, the action is invariant up to a boundary term

$$S^0_{\xi_D}[\Phi,g_{\mu\nu}]=S^0_{\xi_D}[\bar{\Phi},\bar{g}_{\mu\nu}]+\text{b.t.}, \tag{4.3}$$

where we denote as S^0_ξ the part of action (1.1) without the self-interaction contribution. A consequence of this symmetry is that in the conformal frame defined by the particular choice of the conformal factor $\Omega=(\Phi/\Phi_0)^{2/(D-2)}$, the scalar field turns into a constant value Φ_0 and the action becomes proportional to the Einstein-Hilbert action. At the level of the field equations, this translates to the fact that solutions of the stealth equations (1.2) in the case of the conformal coupling (3.14) map to solutions of the vacuum Einstein equations. As we have shown in the previous section, the general solution of the stealth constraints (1.2) defined on a Lifshitz spacetime (2.2) with nontrivial dynamical exponent z is given by (3.12). Then, it is easy to realize that for the dynamical critical exponent $z=2(D-2)/(D-3)$, the Lifshitz stealth becomes a conformally invariant configuration since in this case $\xi_{\rm L}=\xi_D$. Additionally, at the mentioned conformal frame the vacuum metric conformally related to the Lifshitz one exhibits hyperscaling violation with exponent $\theta=(D-1)(D-2)/(D-3)$. Hence, we have justified the existence of General Relativity vacua for the hyperscaling violation metric with exponents given by (4.1), but using a very simple and elegant conformal argument. Note that the holographic character of the Lifshitz stealth is behind this mapping. Indeed, if the scalar field would have depended on some other coordinates, this mapping to the hyperscaling violation metric would not have been possible.

A less known fact is the change of the action (1.1) for a non-conformal coupling $\xi\neq\xi_D$ under a conformal transformation (4.2). Actually, in a generalized conformal frame defined by $\Omega=(\Phi/\Phi_0)^\alpha$, remarkably, the action without potential becomes proportional to the same action but with a different nonminimal coupling. More precisely, the following 1-parameter field transformations

$$\bar{g}_{\mu\nu}=\left(\frac{\Phi}{\Phi_0}\right)^{2\alpha}g_{\mu\nu},\qquad \bar{\Phi}=\Phi_0\left(\frac{\Phi}{\Phi_0}\right)^{[2-(D-2)\alpha]/2}, \tag{4.4a}$$

together with the reparameterization

$$\bar{\xi} = \frac{1}{4}\frac{[(D-2)\alpha-2]^2\,\xi}{1+\alpha\xi(D-1)\,[(D-2)\alpha-4]}, \tag{4.4b}$$

define a map between the actions of two massless free scalar fields with nonminimal couplings given by ξ and $\bar{\xi}$. Concretely, we have

$$\frac{1}{\xi}\,S^0_\xi[\Phi, g_{\mu\nu}] = \frac{1}{\bar{\xi}}\,S^0_{\bar{\xi}}[\bar{\Phi}, \bar{g}_{\mu\nu}] + \text{b.t.} \tag{4.5}$$

Notice that when both nonminimal couplings coincide the transformation (4.4) obviously becomes a symmetry of the nonminimally coupled action, which is nothing but the conformal symmetry since this occurs only if $\bar{\xi} = \xi = \xi_D$.

We will exploit the above conformal argument to build a stealth configuration on the hyperscaling violation metrics (2.3) using as seed those existing on the Lifshitz background (2.2) and defined by (3.12). The generalized conformal frame defined by $\Omega = (\Phi/\Phi_0)^\alpha$ will correspond to the hyperscaling violation metric (2.3) only if $\alpha = 2\theta/[(D-2)(z+D-2)]$. This in turn implies that the configuration given by

$$\xi_{\rm H} \equiv \frac{1}{4}\frac{(D-2)(\theta-z-D+2)^2}{(D-1)(\theta-z-D+2)^2+(D-3)z^2-2(D-2)z}, \tag{4.6a}$$

$$\Phi(r) = \Phi_0\left(\frac{l}{r}\right)^{\frac{z+D-2-\theta}{2}}, \tag{4.6b}$$

defines a stealth on the hyperscaling violation metric (2.3). Consistently, the limit $\theta \to 0$ reduces to the stealth solution on the Lifshitz background (3.12). We can also observe that for the values of the exponents which solve the vacuum Einstein equations (4.1), the stealth scalar field (4.6b) becomes precisely a constant and the hyperscaling violating nonminimal coupling (4.6a) becomes just the conformal one (3.14). In fact, $\xi_{\rm H} = \xi_D$, only if the dynamical exponent takes the value (4.1) analyzed at the beginning of the subsection or if it vanishes. This last case will be studied at the end of the next subsection since it entails more general configurations than the holographic ones, due to the emergence of conformal symmetry for $z = 0$.

4.2 Non-holographic branches: Full derivation

The conformal argument establishing the existence of stealth configurations for spacetimes with hyperscaling violation is undoubtedly a very elegant one. Unfortunately, it lacks the skill to exclude the existence of more general configurations, in particular non-holographic ones. Before, we just establish that for a generic value of the hyperscaling violation exponent θ a stealth solution can be conformally generated, which is holographic by construction. However, as shown below there also exist two special families for particular values for θ, parameterized in terms of the dynamical critical exponent z for any dimension, which give

rise to stealth configurations that are not holographic. These latter are additionally self-interacting in contrast to the free behavior of the holographic one obtained from the previous conformal mapping. Moreover, as in the Lifshitz situation, the conformally flat case $z = 0$ deserves a separate analysis that we will perform at the end of the subsection. Hence, most of this subsection is restricted to the study of anisotropic backgrounds with nontrivial dynamical exponents, $z \neq 1$ and $z \neq 0$.

In order to make transparent the previously anticipated conclusion we shall make use of two nontrivial linear combinations of the stealth constraints (1.2) together with their derivatives, all appropriately evaluated on the hyperscaling violating spacetime (2.3). We start studying the off-diagonal stealth constraints, where we make use again of the redefinition (3.1), and we yield to the following separation

$$\sigma(t,r,x^i) = \left(\frac{l}{r}\right)^{\frac{\theta}{D-2}} \left[\left(\frac{r}{l}\right)^z T(t) + \frac{l}{r}H(r) + \frac{r}{l}\left[X^1(x^1) + \cdots + X^{D-2}(x^{D-2})\right]\right], \tag{4.7}$$

where the unknown functions are again defined modulo the same residual symmetries characterizing the Lifshitz case (3.6). Additionally, the differences between the diagonal spatial components once more establish the conditions (3.8). The first nontrivial combination we now use is given by

$$\begin{aligned}
&\left(\frac{l}{r}\right)^{\frac{\theta}{D-2}} \frac{\sigma}{\Phi^2}\left[\frac{(D-1)(\theta-z-D+2)(\xi-\xi_D)+z\,\xi}{(D-2)\xi}\left(T_t{}^t - T_{(i)}{}^{(i)}\right) - (z-1)\left(T_r{}^r - T_{(i)}{}^{(i)}\right)\right] \\
&\quad + r\partial_r\left[\left(\frac{l}{r}\right)^{\frac{\theta}{D-2}} \frac{\sigma}{\Phi^2}\left(T_t{}^t - T_{(i)}{}^{(i)}\right)\right] = \frac{(z-1)(\theta-z-D+2)^2(\xi-\xi_{\mathrm{H}})}{4l^2\xi_{\mathrm{H}}}\left(\frac{r}{l}\right)^{\frac{\theta}{D-2}}\sigma \\
&\qquad - \frac{4\xi\left[(D-1)(\theta-z-D+2)(\xi-\xi_D)-(D-3)z\xi\right]}{(D-2)(1-4\xi)}\left[\frac{\mathrm{d}^2T}{\mathrm{d}t^2}\left(\frac{r}{l}\right)^{-z} + \frac{\mathrm{d}^2X^i}{\mathrm{d}(x^i)^2}\frac{l}{r}\right] = 0.
\end{aligned} \tag{4.8}$$

The first conclusion we can draw from here is that not only in the holographic case the nonminimal coupling is restricted to take the value $\xi = \xi_{\mathrm{H}}$, but also this restriction must be satisfied for any potentially non-holographic configuration. This can be easily viewed by taking the derivative with respect to any non-holographic coordinate of the right-hand side of (4.8). Notice that these conclusions would be different if $z = 1$ or $z = 0$, and this the reason why these cases deserve a separate attention. We now consider the following second

nontrivial combination evaluated at $\xi=\xi_{\rm H}$

$$\frac{(1-4\,\xi_{\rm H})}{4{\xi_{\rm H}}^2}\left(\frac{l}{r}\right)^{\frac{\theta}{D-2}}\frac{\sigma}{\Phi^2}\left((z-1)T_r{}^r-zT_t{}^t+T_{(i)}{}^{(i)}\right)$$

$$=\frac{z(z-1)\{[(D-3)z-D+2]\theta-(D-2)^2(z-1)\}[\theta-(D-2)(z-1)]}{(D-2)^2(\theta-z-D+2)^2l^2}\left(\frac{r}{l}\right)^{\frac{\theta}{D-2}}\sigma$$

$$+\,z\frac{\mathrm{d}^2T}{\mathrm{d}t^2}\left(\frac{r}{l}\right)^{-z}+\frac{\mathrm{d}^2X^i}{\mathrm{d}(x^i)^2}\frac{l}{r}+\frac{z-1}{lr}\left(r\frac{\mathrm{d}}{\mathrm{d}r}-2\right)\left(r\frac{\mathrm{d}}{\mathrm{d}r}-(z+1)\right)H=0. \tag{4.9}$$

Again, taking the derivative with respect to any non-holographic coordinate, we arrive to the conclusion that non-holographic stealth configurations are only allowed for the following two values of the hyperscaling violating exponent

$$\theta=\frac{(D-2)^2(z-1)}{(D-3)z-D+2}, \tag{4.10a}$$

$$\theta=(D-2)(z-1). \tag{4.10b}$$

One also concludes that not only the second derivatives of the functions holding the spatial dependence are constant (3.8) but also the second derivative of the function that encloses the time evolution. Returning to the former restriction (4.8) and considering that $\xi=\xi_{\rm H}$ together with the previous values for the hyperscaling violating exponent allowing non-holographic behaviors, it is easy to realize that the coefficient in front of these constant second derivatives at (4.8) never vanishes; hence the second derivatives of the functions enclosing the non-holographic dependencies are all necessarily zero, which entails only linear non-holographic regimes for the redefinition of the stealth scalar field (3.1).

For hyperscaling violating exponents different from the values (4.10) the stealth must be necessarily holographic. An exhaustive study of those cases just give rises to the stealth conformally constructed from the Lifshitz one and derived in the previous subsection (4.6).

The hyperscaling violating exponents (4.10) are mutually exclusive in an anisotropic context ($z\neq1$) since the resulting nonminimal coupling (4.6a) must be nontrivial in order for any stealth configuration to be possible; in fact, the vacuum related exponents (4.1) are the only anisotropic possibility. Actually, the exponents (4.10) are not only different numerically but, as discussed below, they also characterize qualitatively different non-holographic behaviors.

Let us start analyzing the first case where the hyperscaling violating exponent takes the value (4.10a). Evaluating the next combination at this value we obtain

$$\frac{(1-4\,\xi_{\rm H})}{4{\xi_{\rm H}}^2}\left(\frac{l}{r}\right)^{\frac{\theta}{D-2}}\frac{\sigma}{\Phi^2}\left(T_t{}^t-T_{(i)}{}^{(i)}\right)=\frac{z-1}{l^2}\left[\frac{l}{r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}-(z+1)\right)H-(z-1)\frac{r}{l}\sum_{j=1}^{D-2}X^j\right]=0. \tag{4.11}$$

Taking the derivative with respect to the spatial coordinates of the previous combination it is possible to conclude that the spatial functions are all constants, which can be taken to be zero

using the residual symmetry (3.6b) of the separability ansatz (4.7). The resulting equation easily fixes the holographic dependence and since it is just a first integral of Eq. (4.9) this last one is now completely satisfied. All the stealth constraints are satisfied except the one that fixes the self-interaction which turns to be a power law of the scalar field. Finally, the first class of non-holographic stealth configurations defined in the hyperscaling violation metric is given by

$$\theta = \frac{(D-2)^2(z-1)}{(D-3)z-D+2}, \tag{4.12a}$$

$$\xi_{\,\mathrm{Ho}} \equiv \frac{z[(D-3)z-2(D-2)]}{4[2(D-3)z^2-4(D-2)z+D-2]}, \tag{4.12b}$$

$$\Phi(x^\mu) = \left[\left(\frac{l}{r}\right)^{\frac{z(D-2)-\theta}{D-2}} \frac{1}{k_t t+\sigma_0}\right]^{\frac{2\,\xi_{\,\mathrm{Ho}}}{1-4\,\xi_{\,\mathrm{Ho}}}}, \tag{4.12c}$$

$$U(\Phi) = \frac{2\,{\xi_{\,\mathrm{Ho}}}^2}{(1-4\,\xi_{\,\mathrm{Ho}})^2}\,\lambda\,\Phi^{(1-2\,\xi_{\,\mathrm{Ho}})/\xi_{\,\mathrm{Ho}}}, \qquad \lambda = -k_t^2. \tag{4.12d}$$

Note that for $k_t = 0$, we just recover the holographic stealth (4.6) for the hyperscaling violating exponent (4.12a). For $k_t \neq 0$, the constant σ_0 can be put to zero by a time translation due to the stationarity of the hyperscaling violating metric (2.3). We emphasize that the self-interaction (4.12d) is negative definite since the involved coupling constant λ must be strictly negative. This overflying stealth is not characterized by any free integration constant.

Let us now consider the other option characterized by a hyperscaling violating exponent with value (4.10b). In this case the same combination after being evaluated at this exponent gives

$$\frac{(1-4\,\xi_{\,\mathrm{H}})}{4{\xi_{\,\mathrm{H}}}^2}\left(\frac{l}{r}\right)^{\frac{\theta}{D-2}}\frac{\sigma}{\Phi^2}\left(T_t^{\ t} - T_{(i)}^{\ (i)}\right) = \frac{z-1}{l^2}\left[\frac{l}{r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}-2\right)H + (z-1)\left(\frac{r}{l}\right)^z T\right] = 0. \tag{4.13}$$

We conclude that the temporal dependence is constant and can be chosen to be zero using the residual symmetry (3.6c). From here, we find the holographic dependence, which is also compatible with Eq. (4.9) since (4.13) becomes again its first integral. Finally, along the same lines as before, the second non-holographic configuration reads

$$\theta = (D-2)(z-1), \tag{4.14a}$$

$$\xi_{\,\mathrm{Hi}} \equiv \frac{(D-3)z-2(D-2)}{4[(D-2)z-2(D-1)]}, \tag{4.14b}$$

$$\Phi(x^\mu) = \left[\left(\frac{r}{l}\right)^{z-2} \frac{1}{\vec{k}\cdot\vec{x}+\sigma_0}\right]^{\frac{2\,\xi_{\,\mathrm{Hi}}}{1-4\,\xi_{\,\mathrm{Hi}}}}, \tag{4.14c}$$

$$U(\Phi) = \frac{2\,{\xi_{\,\mathrm{Hi}}}^2}{(1-4\,\xi_{\,\mathrm{Hi}})^2}\,\lambda\,\Phi^{(1-2\,\xi_{\,\mathrm{Hi}})/\xi_{\,\mathrm{Hi}}}, \qquad \lambda = \vec{k}^2. \tag{4.14d}$$

Note that if $\vec{k}=0$, this solution becomes the holographic stealth (4.6) when the hyperscaling violating exponent is given by (4.14a). Instead, if $\vec{k}\neq 0$, the constant σ_0 can be assumed again as vanishing using the invariance under spatial translation exhibited by the hyperscaling violating metric (2.3). This metric is additionally invariant under spatial rotation, which allows to fix the inhomogeneity along any preferred direction eliminating all the components of the vector $\vec{k}$ except one, which is related to the coupling constant λ (4.14d). Hence, no independent integration constant characterizes this inhomogeneous stealth as in the previous case but with the difference that now the self-interaction is positive definite.

As was emphasized in this analysis, the above conclusions are valid in the anisotropic case $z\neq 1$ and for nonvanishing dynamical exponents $z\neq 0$. We end this subsection by examining what is special about the case $z=0$. For Lifshitz backgrounds this case was considered in Subsec. 3.2 and represents a conformally flat spacetime. Due to the conformal relation with the Lifshitz backgrounds which defines the present spacetimes (2.3) the vanishing dynamical exponent also characterizes here a conformally flat spacetime. Additionally, for $z=0$ the hyperscaling violating nonminimal coupling (4.6a) just becomes $\xi_{\mathrm{H}}=\xi_D$, i.e. the conformal coupling (3.14). Contrary to the cases previously studied in this subsection the holographic behavior is lost now for any value of the hyperscaling violating exponent θ, which induces a self-interaction which is just the conformal one. The emergence of conformal symmetry here also, implies that the corresponding stealth is just a conformal transformation of the conformal stealth of the Lifshitz spacetime (3.15)

$$\Phi_{\mathrm{H}}=\left(\frac{r}{l}\right)^{\frac{\theta}{2}}\Phi_{\mathrm{L}}, \tag{4.15}$$

with a conformal potential whose coupling constant is specified again as in (3.15b). This is the only nontrivial situation allowed for $z=0$. We emphasize that when additionally $\theta=D-2$ this background becomes precisely flat spacetime, whose stealths were studied in Ref. [3] and exist for any value of the nonminimal coupling parameter. Finally, we mention by completeness that other case not studied here is $z=1$, this case also represents a conformally flat spacetime since the background is conformal to AdS, hence a conformally generated solution as the previous one also exists. The isotropy of this case makes it more rich since it additionally allows a different class of holographic configurations valid for any value of the nonminimal coupling, we do not bring the related details here because the emphasis of this work is in anisotropy.

4.3 Nonminimal coupling $\xi=1/4$

We end this section by analyzing the case $\xi=1/4$ which is outside of the previous derivation since we have assumed the redefinition (3.1). In the present situation, it is more pertinent to redefine the scalar field as

$$\Phi=\frac{1}{\sqrt{\kappa}}\,\mathrm{e}^{\sigma}. \tag{4.16}$$

First, notice that contrary to the Lifshitz nonminimal coupling (3.12a), the hyperscaling violating nonminimal coupling (4.6a) is not bounded from above; hence, in principle, it can

achieve the value $\xi_{\rm H} = 1/4$. In fact, a careful study of holographic stealths with coupling $\xi = 1/4$ on these spacetimes just reproduces the stealth (4.6) conformally generated from the Lifshitz one when $\xi_{\rm H} = 1/4$, which is obviously obtained for a couple of values of the exponent θ satisfying a resulting quadratic equation.

Something similar occurs with the nonminimal couplings $\xi_{\rm Ho}$ and $\xi_{\rm Hi}$ allowing the nonholographic behaviors (4.12) and (4.14), respectively, since both are compatible with the hyperscaling violating nonminimal coupling (4.6a) and consequently can reach the value 1/4. The related configurations can be obtained from (4.12) and (4.14) via the nontrivial limit outlined at the end of Subsec. 3.2. In order to obtain well-behaved limits we need to redefine the involved integration constants according to

$$k_\mu = -\frac{1-4\xi}{2\xi}\hat{k}_\mu, \qquad \sigma_0 = 1 - \frac{1-4\xi}{2\xi}\ln\left(\sqrt{\kappa}\,\Phi_0\right), \tag{4.17}$$

and use the appropriated value of the nonminimal coupling in each case.

Starting from the first example (4.12) the value $\xi_{\rm Ho} = 1/4$ is achieved for the two dynamical exponents $z_\pm$ defined below, which reduce the hyperscaling violating exponent of the solution to $\theta = (D-2)z_\pm$. Taking the nontrivial limit $z \to z_\pm$ in (4.12), after applying the redefinitions (4.17) evaluated at $\xi = \xi_{\rm Ho}$, we obtain the following configuration which describes a massive free stealth overflying the spacetime

$$z_\pm = \frac{\sqrt{D-2}}{D-3}\left(\sqrt{D-2} \pm 1\right), \tag{4.18a}$$

$$\theta = (D-2)z_\pm, \tag{4.18b}$$

$$\Phi(x^\mu) = \Phi_0\, e^{\hat{k}_t t}\left(\frac{r}{l}\right)^{\pm\frac{\sqrt{D-2}}{2}}, \tag{4.18c}$$

$$U(\Phi) = \frac{1}{2}m^2\,\Phi^2, \qquad m^2 = -\hat{k}_t^2. \tag{4.18d}$$

For $\hat{k}_t = 0$, the above configuration reduces to the holographic stealth (4.6) when the exponents take the previous values. If $\hat{k}_t \neq 0$, the constant Φ_0 can be tuned using time translations, and the free fields have a tachyonic behavior. Again, these stealths are free of any independent integration constant.

Finally, starting from the second example (4.14) the coupling becomes $\xi_{\rm Hi} = 1/4$ only if $z = 2$, this implies that the other exponent must take the value $\theta = D-2$. Now considering the nontrivial limit $z \to 2$ in (4.14), after using the redefinitions (4.17) for $\xi = \xi_{\rm Hi}$, we obtain the following inhomogeneous massive configurations

$$z = 2, \tag{4.19a}$$

$$\theta = D-2, \tag{4.19b}$$

$$\Phi(x^\mu) = \Phi_0\,\frac{l}{r}\exp\left(\vec{\hat{k}}\cdot\vec{x}\right), \tag{4.19c}$$

$$U(\Phi) = \frac{1}{2}m^2\,\Phi^2, \qquad m^2 = \vec{\hat{k}}^2. \tag{4.19d}$$

Once more, if $\vec{\hat{k}}=0$ the configuration becomes the holographic stealth (4.6) for the present exponents while for $\vec{\hat{k}}\neq 0$ we fix the constant Φ_0 by translations, the direction of the vector $\vec{\hat{k}}$ by rotations and its length by the mass m. Consequently, no independent integration constant characterizes this stealth.

All these are precisely the unique solutions which are obtained by straightforwardly integrating the stealth constraints (1.2) on the hyperscaling violating background (2.3) for the nonminimal coupling $\xi=1/4$.

5. Stealths on Schrödinger backgrounds

In this section we analyze the consequences on the existence of stealths for the spacetime realization of anisotropic scaling, without and with hyperscaling violation, but this time using as gravitational duals the backgrounds (2.4) and (2.5), inspired by the symmetries of the Schrödinger equation for a free particle. Since the methods are similar to those used in the Lifshitz cases, here we will only mention and discuss the different allowed cases without including their deduction.

5.1 Schrödinger inspired dynamical scaling

Following similar arguments to those described in the Lifshitz case and using the results obtained in [18, 19], we can derive the most general solutions of the stealth constraints (1.2) on a Schrödinger background (2.4). First, for a nontrivial value of the dynamical critical exponent $z\neq 1$ and $z\neq 0$ it can be concluded again that stealth configurations are only possible on the Schrödinger background for a precise nonminimal coupling parameter

$$\xi_{\mathrm{S}}\equiv\frac{2z+D-3}{4(2z+D-2)}. \tag{5.1a}$$

The isotropic case $z=1$ once again corresponds to AdS space whose stealths were studied in [4] and the vanishing case $z=0$ will be treated separately at the end of the subsection. For a generic value of the exponent the stealth configuration is self-interacting and determined by

$$\Phi(x^\mu)=\left(\frac{y}{l}\frac{1}{k_u u+\vec{k}\cdot\vec{x}+\sigma_0}\right)^{\frac{2z+D-3}{2}}, \tag{5.1b}$$

$$U(\Phi)=\frac{2\,\xi_{\mathrm{S}}\Phi^2}{(1-4\xi_{\mathrm{S}})^2}\left(\xi_{\mathrm{S}}\lambda\Phi^{\frac{1-4\xi_{\mathrm{S}}}{\xi_{\mathrm{S}}}}+\frac{4\,D\,(D-1)}{l^2}(\xi_{\mathrm{S}}-\xi_D)(\xi_{\mathrm{S}}-\xi_{D+1})\right),\qquad \lambda=\vec{k}^2. \tag{5.1c}$$

For $k_u=0=\vec{k}$, the solution is a holographic massive free stealth with mass

$$m_{\mathrm{S}}^2=\frac{2(z-1)(2z-1)\xi_{\mathrm{S}}}{l^2}, \tag{5.2}$$

however, this is far from being the more general situation. If $\vec{k}\neq 0$, the solution is in fact stationary since time dependence can be eliminated via a Galilean boost which is one of the

symmetries underlying the Schrödinger metric (2.4), see Ref. [20]. This explains why the coupling constant λ is independent of the constant k_u. Using the invariance under spatial rotations the vector $\vec{k}$ can be aligned along one of the spatial directions x^i. In turn, the constant σ_0 can be eliminated exploiting the translation invariance (along space or time). In summary, there are no independent integration constants if $\vec{k} \neq 0$. For $\vec{k} = 0$ and $k_u \neq 0$, the solution describes a time-dependent massive free stealth overflying the Schrödinger background with the same mass (5.2). Notice that the constant k_u can be tuned to any predetermined value using the dynamical scaling symmetry of the Schrödinger spacetime, and this case is also free of any independent integration constants.

The above configuration becomes enhanced for the dynamical scaling $z = 1/2$ due to the emergence of conformal symmetry. Indeed, for $z = 1/2$ the Schrödinger spacetimes have the special property that their Weyl tensor vanishes, i.e. they turn into conformally flat spacetimes for this value. This is manifest by rewriting the metric (2.4) for $z = 1/2$ as

$$\begin{aligned} ds_{\mathrm{S}}^2 &= \frac{l^2}{y^2}\left(-\frac{y}{l}du^2 - 2dudv + dy^2 + d\vec{x}^2\right) \\ &= \frac{l^2}{y^2}\left\{-2\,du\,d\left(v + \frac{u^3}{24l^2} + \frac{uy}{2l}\right) + \left[d\left(y + \frac{u^2}{4l}\right)\right]^2 + d\vec{x}^2\right\} \equiv \Omega^2 \eta_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu, \end{aligned} \tag{5.3}$$

where $\{\bar{x}^\mu\}$ are the standard cartesian coordinates of flat spacetime, defined here from the above light-cone representation. The enhancement occurs because the Schrödinger nonminimal coupling (5.1a) becomes the conformal one (3.14) for $z = 1/2$, that is $\xi_{\mathrm{S}} = \xi_D$, which in turn causes that the conformally invariant potential also emerges from self-interaction (5.1c). In other words, the stealth action (1.1) becomes conformally invariant and having a concrete example of stealth configuration implies that its whole conformal class also allows a stealth interpretation. Hence, due to the conformally flat character of the Schrödinger spacetimes for $z = 1/2$, their stealth configurations in this case can be obtained from a conformal transformation of the stealths defined on flat spacetime [3] (a similar mechanism works for the conformal stealths allowed for any standard cosmology [5]). More concretely, the conformal transformation between the $z = 1/2$ Schrödinger stealths Φ_{S} and the flat configurations Φ_{F} [3], inferred from the conformal relation (5.3), is

$$\begin{aligned} \Phi_{\mathrm{S}}(x^\mu) &= \Omega^{-(D-2)/2}\,\Phi_{\mathrm{F}}(\bar{x}^\mu) = \left[\Omega\left(\frac{\alpha}{2}\eta_{\mu\nu}\bar{x}^\mu\bar{x}^\mu + k_\mu\bar{x}^\mu + \sigma_0\right)\right]^{-(D-2)/2} \\ &= \left(\frac{l}{y}\left\{\frac{\alpha}{2}\left[-2\,u\left(v + \frac{uy}{2l} + \frac{u^3}{24l^2}\right) + \left(y + \frac{u^2}{4l}\right)^2 + \vec{x}^2\right] + k_u u + k_v\left(v + \frac{uy}{2l} + \frac{u^3}{24l^2}\right)\right.\right. \\ &\qquad \left.\left. + k_y\left(y + \frac{u^2}{4l}\right) + \vec{k}\cdot\vec{x} + \sigma_0\right\}\right)^{-(D-2)/2}. \end{aligned} \tag{5.4a}$$

Remarkably, explicitly solving the stealth constraints (1.2) for $z = 1/2$ Schrödinger spacetimes, in a lengthy process, gives exactly the same result. The relation between the conformal

coupling constant and the resulting integration constants is the same than in flat spacetime [3]

$$U(\Phi)=\frac{(D-2)^2}{8}\lambda\Phi^{\frac{2D}{D-2}},\qquad \lambda=-2k_uk_v+k_y^2+\vec{k}^2-2\alpha\sigma_0. \tag{5.4b}$$

In flat spacetime and for $\alpha\neq 0$ one can use the translation invariance to put the constants k_μ to zero all ; consequently, the conformal stealth depends on a single independent integration constant (the other one is determined by the coupling constant), related to the existence of the conformal symmetry. In the Schrödinger case the translation invariance along the coordinate y is lost and we can only choose k_u, k_v and $\vec{k}$ to be zero. However, the constant k_y is not arbitrary since it can be fixed by an anisotropic scaling. Consequently, the conformal stealth of $z=1/2$ Schrödinger spacetime has again a single independent integration constant as its conformal cousin of flat spacetime. This is not always the case for conformally flat configurations, as it is evidenced by the cosmological configurations [5], where usually the conformal factor breaks translation invariance increasing the number of independent integration constants. The difference here is that this breaking is compensated by the existence of the scaling symmetry. Something similar occurs for the isotropic case $z=1$ which is also conformally flat defining AdS space [4]. For $\alpha=0$ and $\vec{k}\neq 0$ then k_u and σ_0 can be put to be zero by a Galilean boost and a translation, respectively. The constant k_v becomes fixed by a scaling, and a rotation is responsible of aligning the vector $\vec{k}$ along a particular spatial direction. Consequently, the solution again depends on a single independent integration constant. For $\alpha=0=\vec{k}$, if $k_u\neq 0$ or $k_v\neq 0$ then one of them is fixed by a scaling and σ_0 can be chosen to be zero by a translation. The solution would depend on a single independent integration constant if both k_u and k_v are nonvanishing, and also if both are vanishing (since σ_0 would remain free). If only one of them vanishes there is no independent integration constant at all. This is also the case for $\alpha=0=k_\mu$, which coincides just with the $z=1/2$ holographic massless free stealth solution (5.1).

We mentioned at the beginning that the dynamical exponent $z=0$ is exceptional in the sense that the nonminimal coupling is not necessarily restricted in this case. That is, for $z=0$ the solution (5.1) is still valid and the Schrödinger nonminimal coupling becomes $\xi_{\mathrm{S}}=\xi_{D-1}$, however, the following additional branch occurs if the nonminimal coupling is allowed to take any other different value

$$z=0, \tag{5.5a}$$

$$\Phi(x^\mu)=\left[\frac{y}{l}\left(\frac{k_u}{\omega}\sin(\omega u)+\sigma_0\cos(\omega u)\right)^{-1}\right]^{\frac{2\xi}{1-4\xi}},\qquad \omega^2=\frac{(D-2)(\xi-\xi_{D-1})}{\xi l^2}, \tag{5.5b}$$

$$U(\Phi)=\frac{8\,D\,(D-1)\xi(\xi-\xi_D)(\xi-\xi_{D+1})}{(1-4\xi)^2l^2}\,\Phi^2. \tag{5.5c}$$

In the limit $\xi\to\xi_{D-1}$ the frequency ω vanishes and we recover the solution (5.1) with $\vec{k}=0$. Here, the constant σ_0 can be eliminated by a time translation and the constant k_u can be

fixed by a scaling (since the retarded time u remains unchanged for $z = 0$); once again the stealth has no independent integration constants.

Finally, notice that $\xi_{\rm S} < 1/4$ in the solution (5.1) and in the limit $\xi \to 1/4$ there is no redefinition allowing the solution (5.5) to have the well-behaved behavior defined in (3.17); the value $\xi = 1/4$ is excluded from the presented solutions. A careful study of this case shows that like in the other paradigmatic example of anisotropic background, i.e. the Lifshitz spacetime, no stealth configuration on the Schrödinger spacetime exists for the nonminimal coupling $\xi = 1/4$ if one consider a generic value of the dynamical exponent z. But, contrary to the Lifshitz example, there is no exceptional anisotropic value of the exponent allowing solutions for this coupling.

5.2 Schrödinger inspired hyperscaling violation

As in the standard hyperscaling violation case (2.3), we would like to exploit the obvious conformal relation between the Schrödinger background (2.4) and those exhibiting hyperscaling violation inspired by the Schrödinger line element (2.5). Hence, we start by using a conformal argument to map stealth configurations on both backgrounds. However, as we have emphasized previously, in order to perform this task it is vital for the scalar field to be a power of the conformal factor depending, this time, exclusively on the coordinate y. Since, unlike the Lifshitz case, the Schrödinger stealths are not necessarily holographic, these configurations are obtained by imposing the conditions $\alpha = 0 = k_\mu$ in the first two examples (5.1) and (5.4). They describe in general holographic massive free stealths with mass (5.2). In order to use the conformal mapping, the first step is to show that the transformations (4.4) between two massless free nonminimally coupled actions (4.5), also relate the actions when self-interactions are present. Concretely, the formula (4.5) can be extended to

$$\frac{1}{\xi}\left(S^0_\xi[\Phi, g_{\mu\nu}] - \int d^Dx\sqrt{-g}\,U(\Phi)\right) = \frac{1}{\bar{\xi}}\left(S^0_{\bar{\xi}}[\bar{\Phi}, \bar{g}_{\mu\nu}] - \int d^Dx\sqrt{-\bar{g}}\,\bar{U}(\bar{\Phi})\right) + \text{b.t.}, \tag{5.6a}$$

where the self-interactions must be related by

$$\bar{U}(\bar{\Phi}) = \frac{\bar{\xi}}{\xi}\left(\bar{\Phi}/\Phi_0\right)^{\frac{2D\alpha}{(D-2)\alpha-2}}\,U\Big(\Phi_0\left(\bar{\Phi}/\Phi_0\right)^{\frac{2}{2-(D-2)\alpha}}\Big)\,. \tag{5.6b}$$

For example, starting with a potential given by a superposition of power laws of the scalar field as $U(\Phi) = \sum_i \lambda_i \Phi^{\sigma_i}$, the new potential results again in a superposition of power laws of the new scalar field where the new parameters are defined by

$$\bar{\lambda}_i = \lambda_i \frac{\bar{\xi}}{\xi}\Phi_0^{\frac{\alpha(D-2)\sigma_i - 2\alpha D}{\alpha(D-2)-2}}, \qquad \bar{\sigma}_i = \frac{2(\alpha D - \sigma_i)}{\alpha(D-2)-2}. \tag{5.7}$$

Applying this last version of the map to the holographic massive free configurations (5.1), a straightforward computation shows that the holographic self-interacting configuration given

by

$$\xi_{\rm HS} \equiv \frac{1}{4}\frac{(D-2)(\theta-2z-D+3)^2}{(D-1)(\theta-2z-D+3)^2-(2z-1)(2z+D-3)}, \tag{5.8a}$$

$$\Phi(y) = \left[\left(l\sqrt{\lambda}\right)^{\frac{D-2}{\theta}}\frac{y}{l}\right]^{\frac{2z+D-3-\theta}{2}}, \tag{5.8b}$$

$$U(\Phi) = (z-1)(2z-1)\,\xi_{\rm HS}\,\lambda\,\Phi^{\frac{2[D\theta-(D-2)(2z+D-3)]}{(D-2)(\theta-2z-D+3)}}, \tag{5.8c}$$

satisfies the stealth constraints (1.2) on the backgrounds exhibiting hyperscaling violation *à la* Schrödinger (2.5). The constant appearing in the transformation (4.4) is naturally defined here in terms of the integration constant of the starting holographic configuration (5.1) as $\Phi_0^2 = \sigma_0^{-(2z+D-3)}$. This constant can be fixed arbitrarily on the Schrödinger background using a scaling, however, this is no longer the case in the hyperscaling violation context since now it defines the coupling constant of the self-interaction (5.8c) via $\Phi_0^2 = (l\sqrt{\lambda})^{(D-2)(2z+D-3-\theta)/\theta}$. We would like to stress that the utility of the above approach, beyond its succinctness and beauty, is that if one explicitly solves the stealth constraints (1.2) for generic values of the exponents characterizing these backgrounds the only configuration valid for all the cases is precisely the above. As in the standard hyperscaling violation case, this is not the end of the story since there are also special values of the exponents for which the behavior of the stealths can be different from this conformally generated holographic configuration.

We start the covering of especial cases by pointing out that there are only two special values of the dynamical exponent z for which the Schrödinger hyperscaling violation nonminimal coupling (5.8a) becomes the conformal one (3.14), $\xi_{\rm SH} = \xi_D$. The first, is again the point $z = 1/2$ as in the purely Schrödinger analysis, and the second is for the exponent $z = -(D-3)/2$. For $z = 1/2$ the self-interaction (5.8c) vanishes, but nonholographic contributions emerge in this case and supplement the self-interaction precisely with the conformal potential; as a direct consequence, the stealth action (1.1) is again conformally invariant. Due to the conformal relation of these hyperscaling violation backgrounds to the Schrödinger one (2.5), the resulting conformal configuration is just a conformal transformation of the Schrödinger conformal stealth (5.4)

$$\Phi_{\rm HS} = \left(\frac{l}{y}\right)^{\frac{\theta}{2}}\Phi_{\rm S}, \tag{5.9}$$

where the conformal coupling constant is determined as in Eq. (5.4b). These solutions allow all the subcases already characterized for the Schrödinger example in the paragraph following Eq. (5.4b). For the exponent $z = -(D-3)/2$ not only the Schrödinger hyperscaling violation nonminimal coupling (5.8a) becomes the conformal coupling, but also the self-interaction (5.8c) turns out to be the conformal one. Nothing special happens in this case, since (5.8) just describes a conformal holographic stealth. It is curious that this stealth can be obtained as a conformal transformation of the constant scalar trivial solution allowed by the Schrödinger

AdS-wave

$$ds_{\rm S}^2 = \frac{l^2}{y^2}\left[-\left(\frac{y}{l}\right)^{D-1}du^2 - 2dudv + dy^2 + d\vec{x}^2\right], \tag{5.10}$$

which happens to be the holographic vacuum solution of Einstein equations in the presence of a negative cosmological constant [20], i.e. in this case $\Phi_{\rm HS} = (l/y)^{\frac{\theta}{2}}(l\sqrt{\lambda})^{\frac{2-D}{2}}$.

The other special cases appear when the exponents are taken as $\theta = D-2$ and $z=0$, which reduce the Schrödinger hyperscaling violation backgrounds (2.5) to the pp-wave

$$ds_{\rm HS}^2 = -\left(\frac{y}{l}\right)^2 du^2 - 2dudv + dy^2 + d\vec{x}^2. \tag{5.11}$$

The following purely time-dependent massless free stealth

$$z = 0, \tag{5.12a}$$

$$\theta = D-2, \tag{5.12b}$$

$$\Phi(u) = \left[A\sin\left(\sqrt{\frac{1-4\xi}{4\xi}}\frac{u}{l}\right) + B\cos\left(\sqrt{\frac{1-4\xi}{4\xi}}\frac{u}{l}\right)\right]^{-\frac{2\xi}{1-4\xi}}, \tag{5.12c}$$

overflies the previous pp-wave for a generic value of the nonminimal coupling parameter ξ. There are two special nonminimal couplings for which this purely time-dependent configuration acquires an additional dependence along the holographic direction y. Firstly, for the Schrödinger hyperscaling violation nonminimal coupling (5.8a), which becomes $\xi = \xi_{\rm HS} = 1/8$ for the exponents $\theta = D-2$ and $z = 0$, the additional contribution gives rise to a self-interacting stealth

$$z = 0, \tag{5.13a}$$

$$\theta = D-2, \tag{5.13b}$$

$$\xi = \frac{1}{8}, \tag{5.13c}$$

$$\Phi(x^\mu) = \frac{1}{\sqrt{A\sin\left(\frac{u}{l}\right) + B\cos\left(\frac{u}{l}\right) + \sqrt{\lambda}y}}, \tag{5.13d}$$

$$U(\Phi) = \frac{\lambda}{8}\Phi^6. \tag{5.13e}$$

The other case allowing an additional dependence along the holographic direction is for the conformal coupling, $\xi = \xi_D$, since the hyperscaling violation exponent θ is not necessarily restricted in this case and the extra dependence enters via a nontrivial conformal factor

$$z = 0, \tag{5.14a}$$

$$\xi = \xi_D = \frac{D-2}{4(D-1)}, \tag{5.14b}$$

$$\Phi(x^\mu) = \left(\frac{l}{y}\right)^{\frac{\theta-D+2}{2}}\left[A\sin\left(\frac{u}{l\sqrt{D-2}}\right) + B\cos\left(\frac{u}{l\sqrt{D-2}}\right)\right]^{-\frac{D-2}{2}}. \tag{5.14c}$$

In all these time-dependent solutions the constant B can be eliminated by a time translation and these solutions have a single independent integration constant.

Finally, we comment on the nonminimal coupling value $\xi = 1/4$, which deserves special attention. First, we notice that Schrödinger hyperscaling violation nonminimal coupling (5.8a) contains this value, $\xi_{\rm HS} = 1/4$, for

$$z = \frac{\theta^2 - (D-3)[2\theta - (D-2)]}{2(2\theta - D + 2)}. \tag{5.15}$$

Hence, the behavior of the stealths with $\xi = 1/4$ for a generic value of the hyperscaling violation exponent θ is ruled by the holographic configuration (5.8) restricting the dynamical exponent z as above. An exception must be made again for the exponents $\theta = D - 2$ and $z = 0$ where the holographic behavior is broken according to

$$z = 0, \tag{5.16a}$$

$$\theta = D - 2, \tag{5.16b}$$

$$\xi = \frac{1}{4}, \tag{5.16c}$$

$$\Phi(x^\mu) = \exp\left(\frac{u^2}{4l^2} + k_u u + \vec{k}\cdot\vec{x} + \sigma_0\right), \tag{5.16d}$$

$$U(\Phi) = \frac{1}{2}\, m^2\, \Phi^2, \qquad\qquad m^2 = \vec{k}^2. \tag{5.16e}$$

The above massive configurations with $\vec{k} \neq 0$ have no independent integration constants since k_u and σ_0 can be chosen to be zero using translations, and $\vec{k}$ can be rotated to a particular spatial axis where the related component just determines the mass. We found intriguing that contrary to the previous $\xi = 1/4$ solutions we exhibited, this configuration can not be obtained from a solution with generic values of the nonminimal coupling parameter via the nontrivial limit described in Subsec. 3.2. This is possible only in the massless case $\vec{k} = 0$ which is a nontrivial limit of the purely time-dependent solution (5.12). The limit inherited the property that only one constant can be eliminated by time translation, i.e. these stealths overfly the involved background with a single independent integration constant.

6. Conclusions

Here, we have established the existence of stealth configurations given by scalar fields nonminimally coupled to the gravity of some interesting anisotropic backgrounds such as the Lifshitz and Schrödinger metrics as well as their hyperscaling violation extensions. These backgrounds have been proposed recently to extend the ideas underlying the AdS/CFT correspondence to non-relativistic field theory.

We have first shown that Lifshitz stealths with nontrivial dynamical exponent z, because of the anisotropy, are possible only as massless free scalar fields depending exclusively on the

holographic coordinate. Moreover, the nonminimal coupling is not arbitrary but is parameterized in terms of the dynamical exponent z of the Lifshitz spacetime for any dimension. All these features make the Lifshitz stealth configurations different from those existing in the $z = 1$ isotropic (A)dS case [4] and in the Minkowski spacetime [3], since those latter are generally self-gravitating, depend on all the coordinates and exist for all values of the nonminimal coupling parameter. An exception must be made for the vanishing exponent $z = 0$ where Lifshitz spacetime becomes conformally flat and additional time dependencies appear. Interestingly, within this context, we provide for the first time an approach to obtain stealth solutions with a nonminimal coupling $\xi = 1/4$ from the solutions with a generic value of the coupling which usually exclude this case. The procedure seems to give rise to the more general solutions in most of the cases.

We have taken advantage of the fact that the hyperscaling violation metric is conformally related to the Lifshitz one as well as of the holographic character of the Lifshitz stealth configuration to obtain a stealth configuration defined on the hyperscaling violation background. This has been done by mapping the Lifshitz stealth solution using a very simple conformal argument relating the actions of two different nonminimally coupled massless free theories. We have shown that the resulting configuration is just a particular case of the stealth solutions defined on the hyperscaling violation metric since we have also derived the most general stealth configurations which exhibit a non-holographic character for special values of the exponents. In the last section, we have also considered the case of Schrödinger stealth configurations and shown the character not necessarily holographic of the stealth solutions. This difference is essentially due to the presence of a null direction in the Schrödinger metric. These solutions can be made holographic by fixing appropriately the constants and, extending the conformal argument of the previous case to self-interacting actions, we have mapped these Schrödinger holographic stealth into stealth solutions defined on the Schrödinger background with hyperscaling violation. We have scanned also the special values of the exponents for which the stealth behaviors depart from being holographic.

As a natural but highly nontrivial work, it would be interesting to characterize geometrically all the static spacetimes that may support stealth configurations given by a nonminimal scalar field. The spacetime geometries considered in this paper are all of zero mass. A possible extension of the present work is to look for stealth configurations defined on black holes having as asymptotic the studied anisotropic geometries. These spacetimes play an important role since they are the gravitational duals at finite temperature regime of nonrelativistic gauge/gravity duality. Examples of stealths on black holes are known on the BTZ one [2], on the $z = 1$ black hole of New Massive Gravity [22], and for a more general scalar tensor theory in the case of the Schwarzschild metric or for Lifshitz black holes, see [23, 24]. There are even examples of composite stealths on the BTZ black holes [25].

Another interesting task would consist in finding some applications concerning these stealth configurations in the set-up of the non-relativistic version of the AdS/CFT correspondence. A starting point is studying the perturbations of all the configurations we present here. It is possible that most of them are not stealth themselves due to the rigidity of stealth

solutions, i.e. they do not present integration constants or have the smallest possible number of them, causing the perturbations to evolve necessarily departing from the stealth regime. The coupling of stealth perturbations to the ones of the gravitational duals unavoidably would change the behavior of the last ones bringing nontrivial implications for holographic predictions. We hope to explore this interesting issue in the near future.

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